Spherical Universes with Anisotropic Pressure

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Abstract. Einstein's equations are solved for spherically symmetric universes composed of dust with tangential pressure provided by angular momentum, $L(R)$, which differs from shell to shell. The metric is given in terms of the shell label, R, and the proper time, τ, experienced by the dust particles. The general solution contains four arbitrary functions of R - $M(R)$, $L(R)$, $E(R)$ and $\tau_0(R)$. The solution is described by quadratures, which are in general elliptic integrals. It provides a generalization of the Lemaître-Tolman-Bondi solution. We present a discussion of the types of solution, and some examples. The relationship to Einstein clusters and the significance for gravitational collapse is also discussed.

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1. Introduction

In 1939, Einstein ([1]) solved the field equations of general relativity to describe a cluster of particles moving in circular orbits about the centre of the system. The static and spherically symmetric solution was supported by a balance between gravity and the centrifugal force due to the angular momentum of the particles. These Einstein clusters have been studied extensively. Gilbert [2] investigated their stability, Florides [3] took one as a model for the interior of a star and Comer and Katz [4] used them as a model of a 'heat bath' for a black hole. A natural generalization is to consider a similar system which evolves dynamically. This was first considered by Datta [5], who limited his discussion to the case in which all the particles have the same angular momentum. Bondi [6] generalized Datta's model to allow each shell of particles to have a different angular momentum, and discussed the types of behaviour that might occur. The same model was discussed by Evans [7], who investigated the system for both thin heavy shells of particles, and a cloud of dust. None of these authors obtained a general solution for the metric of the spacetime. Datta obtained the solution for the simple case in which the angular momentum is the same for each shell. Bondi considered matching the general solution to an external Schwarzschild metric, but did not find any additional solutions. Evans discussed how the system would evolve from initial data, but again did not obtain any solutions.

In recent years, much work has been done on gravitational collapse, as an investigation of the Cosmic Censorship Hypothesis (see [8] for a recent review and references therein). In this context, Magli [9] examined this model again. He discussed a general class of spherically symmetric solutions to Einstein's equations with vanishing radial pressure. He was able to solve for the metric in terms of mass-area coordinates [10], based on the method used for charged dust collapse by Ori [11]. Using his solution, Harada *et al* [12] investigated naked singularity formation in the model, and gave a particular case in which the metric could be written in terms of elementary functions. This special case was investigated by Harada *et al* [13] and Kudoh [14], and singularity formation in the general case was examined further by Jhingan and Magli [15].

We tackle this problem in a new way, solving for the metric in terms of the time experienced by the dust particles composing the universe. The mass-area approach has the advantage that the solution reduces to a single integral which is in terms of the coordinates used. This is not true in general in the new approach. However, the mass-area solution makes the structure of spacetime unclear, since both these coordinates can be spacelike in general. Moreover, such coordinates are only appropriate to monotonic collapse or expansion, in which the area can be used as a time coordinate. The mass-proper time solution is better suited to studies of model universes, which may have phases of both expansion and collapse, and it is physical, since the time coordinate is understandable, as the time experienced by the dust.

In section 2, we describe the model and derive the equations. In section 3, we solve the equations, present a general discussion of the types of solution, and describe the relation to Einstein clusters. In section 4, we present some examples. Section 5 gives our conclusions.

2. The Model

The spacetime is composed of a large number of dust particles, which can move radially, and have angular momentum about the origin. Spherical symmetry is assumed, so that all quantities are functions only of the time and the radial coordinate. To produce this symmetry, at each point there must be a large number of particles, moving with the same angular momentum, but in every tangential direction. A comoving radial coordinate, R, is used, i.e. particles remain on surfaces $R = \textit{constant}$ for all time. The general spherically symmetric metric is given by the ansatz:

$$\mathrm{d}s^2 = \mathrm{e}^{2\nu(R,t)}\mathrm{d}t^2 - \mathrm{e}^{2\lambda(R,t)}\mathrm{d}R^2 - r^2(R,t)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{1}$$

We assume that the dust only interacts with itself gravitationally, so that each dust particle moves on a geodesic of the spacetime. As a consequence of the geodesic equations, the angular momentum, $L = r^2\sqrt{(\mathrm{d}\theta/\mathrm{d}\tau)^2 + \sin^2\theta\,(\mathrm{d}\phi/\mathrm{d}\tau)^2}$ is a constant for each particle (where τ denotes the proper time for the particle). Spherical symmetry requires that L be the same for each particle on a given shell, i.e. $L = L(R)$ only. The energy momentum tensor for dust is $T^{\alpha\beta} = \rho u^\alpha u^\beta$, where ρ is the proper density of the dust and $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$ is the 4-velocity. This would be $T^{\alpha\beta}$ for this spacetime if the 4-velocity of the dust was well-defined. However, particles are moving in all tangential directions at each point, and so the 4-velocity is not single valued. But, as the particles are assumed not to interact, the overall energy-momentum tensor can be taken as a sum of the individual contributions over all the particles $T^{\alpha\beta} = \sum \rho u^\alpha u^\beta$. The choice of comoving coordinates makes the radial pressure $-T^R_R$ vanish (u^R=0). Spherical symmetry of the overall system requires the two tangential pressures to be equal $T^\theta_\theta = T^\phi_\phi = -p(R,t)$. For a single dust particle, $u_t u^t = \mathrm{e}^{2\nu}(\mathrm{d}t/\mathrm{d}\tau)^2$. From the geodesic equations, $\mathrm{e}^{2\nu}(\mathrm{d}t/\mathrm{d}\tau)^2 = 1 + L^2 r^{-2}$. Since L is conserved, we have $u^\theta u_\theta + u^\phi u_\phi = -L^2/r^2$. The ratio of $u^\theta u_\theta + u^\phi u_\phi$ to $u_t u^t$ is the same for every particle on a shell, and so the energy momentum tensor is given by

$$2p(R,t) = -T^\theta_\theta - T^\phi_\phi = \frac{L^2}{L^2 + r^2}T^t_t = \frac{L^2}{L^2 + r^2}\epsilon(R,t). \tag{2}$$

$\epsilon(R,t)$ is the energy density, T^t_t, which includes a contribution from the kinetic energy of the particles.

Conservation of the energy-momentum tensor, $T^\alpha_{\beta;\alpha}$, yields the two non-trivial equations:

$$\nu'\epsilon - 2\frac{r'}{r}p = 0 \tag{3}$$

$$\dot{\epsilon} + \dot{\lambda}\epsilon + 2\frac{\dot{r}}{r}(\epsilon + p) = 0. \tag{4}$$

In these, $'=(\partial/\partial R)_t$ and $\dot{}=(\partial/\partial t)_R$. Using the convention $R^\alpha_{\beta\gamma\delta}=\Gamma^\alpha_{\beta\delta,\gamma}-...$, $G^{\alpha\beta}=R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R$ for the Einstein curvature tensor and the identification $r=\mathrm{e}^\mu$, the non-trivial Einstein equations are given by:

$$\frac{8\pi G}{c^4}T^t_R=2\mathrm{e}^{-2\lambda}\left((\dot\mu)'+\dot\mu\mu'-\dot\lambda\mu'-\nu'\dot\mu\right)=0 \tag{5}$$

$$\frac{8\pi G}{c^4}T^t_t=\mathrm{e}^{-2\nu}\left(2\dot\lambda\dot\mu+(\dot\mu)^2\right)+\mathrm{e}^{-2\mu}-\mathrm{e}^{-2\lambda}\left(2\mu''+3\left(\mu'\right)^2-2\lambda'\mu'\right)-\Lambda=\frac{8\pi G}{c^4}\epsilon \tag{6}$$

$$\frac{8\pi G}{c^4}T^R_R=\mathrm{e}^{-2\nu}\left(2\ddot\mu+3\left(\dot\mu\right)^2-2\dot\nu\dot\mu\right)+\mathrm{e}^{-2\mu}-\mathrm{e}^{-2\lambda}\left(\left(\mu'\right)^2+2\nu'\mu'\right)-\Lambda=0 \tag{7}$$

$$\begin{aligned}-\frac{8\pi G}{c^4}T^\theta_\theta&=\mathrm{e}^{-2\nu}\left(\dot\lambda\dot\nu+\dot\nu\dot\mu-\dot\lambda\dot\mu-\ddot\lambda-\dot\lambda^2-\ddot\mu-\dot\mu^2\right)\\&\quad+\mathrm{e}^{-2\lambda}\left(\nu''+\nu'^2+\mu''+\mu'^2-\lambda'\mu'-\nu'\lambda'+\nu'\mu'\right)+\Lambda=\frac{8\pi G}{c^4}p.\end{aligned} \tag{8}$$

These are the field equations of Landau and Lifshitz ([16] §100), but we have written 2λ for their λ, 2μ for their μ and 2ν for their ν and included the cosmological constant Λ. In the rest of the paper the areal radius $r=\mathrm{e}^\mu$ will be used. From (2) and (3),

$$\nu'=\frac{L^2}{L^2+r^2}\frac{r'}{r}. \tag{9}$$

Substitution into equation (5) and integration gives

$$\mathrm{e}^{2\lambda}=\frac{r'^2\left(1+\frac{L^2}{r^2}\right)}{1+2E}. \tag{10}$$

In (10), the constant of integration has been taken to be $-\ln(1+2E(R))$. In the Tolman-Bondi case, it is usually denoted as $1+f(R)$. However, the function $1+2E(R)$ should be regarded as the relativistic energy, E_{r}^2, and in the Newtonian limit becomes $1+2E_{\mathrm{N}}(R)$, as will become clear. The use of $E(R)$ instead of f makes this identification more explicit. Substitution of (9) and (10) into (7) and integration gives:

$$\frac{1}{2}\mathrm{e}^{-2\nu}\dot r^2-\frac{GM(R)}{r}-\frac{1}{6}\Lambda r^2+\frac{L^2}{2(L^2+r^2)}=\frac{Er^2}{L^2+r^2}. \tag{11}$$

The constant of integration has been taken to be $-2GM(R)$. On substitution of (9), (10) and (11) into (6) we obtain

$$\frac{2GM'(R)}{r^2r'}=\frac{8\pi G}{c^4}\epsilon\Rightarrow M(R)=\int_0^R4\pi r^2\frac{\epsilon}{c^4}r'\mathrm{d}R. \tag{12}$$

$M(R)$ is the Misner-Sharp mass [17], which is conserved for spherical systems with vanishing radial pressure. This is not the energy density summed over all shells. For that, the factor $exp(\lambda)$ must be included in the integral to give the correct volume element.

3. Mass-Proper Time Solution

The equations (9) and (11) are coupled equations for the two remaining unknown functions, $r(R,t)$ and $\nu(R,t)$. They cannot be solved explicitly in the coordinates t and R. Magli [10] found a solution by using mass-area coordinates, that is r and R. We shall obtain a solution in terms of a different time coordinate. There are two times besides t relevant to this problem. The time experienced by an observer at rest on a shell obeys $\mathrm{d}T^2 = \mathrm{e}^{2\nu}\mathrm{d}t^2$. The equation of motion in T is therefore identical to (11), with the first term replaced by $\frac{1}{2}(\partial r/\partial T)^2_R$. Alternatively, there is the proper time experienced by a dust particle, τ, which obeys

$$\left(\frac{\partial \tau}{\partial t}\right)_R = \frac{\mathrm{e}^{\nu}}{\sqrt{1+\frac{L^2}{r^2}}}. \tag{13}$$

On changing coordinates from (t,R) to (τ,R), equation (11) becomes:

$$\frac{1}{2}\left(\frac{\partial r}{\partial \tau}\right)^2_R = E + \frac{GM}{r}\left(1+\frac{L^2}{r^2}\right) + \frac{1}{6}\Lambda L^2 + \frac{1}{6}\Lambda r^2 - \frac{1}{2}\frac{L^2}{r^2}. \tag{14}$$

This equation is exactly the equation of motion of a test particle with angular momentum L and energy $\sqrt{1+2E}$ in a vacuum Schwarzschild metric of mass M. This result is a consequence of the assumption that the dust particles follow geodesic paths, and justifies the identification of E as the energy and M as the mass made earlier.

Equation (14) may be integrated, keeping R constant:

$$A(r,R) = \int^r \frac{\mathrm{d}r}{(\partial r/\partial \tau)_R} = \tau - \tau_0(R). \tag{15}$$

$\tau_0(R)$ is a constant of integration. The function $A(r,R)$ is given explicitly by (14), since $\partial r/\partial \tau$ is known as a function of r and R. In the case $\Lambda = 0$, the integral is an elliptic integral. For non zero Λ the integral is hyperelliptic. $A(r,R) = \tau - \tau_0(R)$ gives the function $r(R,\tau)$ on inversion. We obtain $(\partial r/\partial R)_\tau$ from (15) as

$$\left(\frac{\partial r}{\partial R}\right)_\tau = -\frac{\left(\frac{\mathrm{d}\tau_0}{\mathrm{d}R} + \left(\frac{\partial A}{\partial R}\right)_r\right)}{\left(\frac{\partial A}{\partial r}\right)_R}. \tag{16}$$

The solution is not yet complete, as the metric has not been written in the new coordinates. To do this, $\mathrm{d}\tau$ must be expressed in terms of $\mathrm{d}R$ and $\mathrm{d}t$. Equation (13) gives $(\partial \tau/\partial t)_R$. To obtain $(\partial \tau/\partial R)_t$, we first differentiate (13) with respect to R and then change the order of partial derivatives, obtaining an expression for $(\partial/\partial t)(\partial \tau/\partial R)_t$. This may be written as an equation for $(\partial/\partial \tau)(\partial \tau/\partial R)_t$ by noting

$$\mathrm{e}^{-\nu}\sqrt{1+\frac{L^2}{r^2}}\left(\frac{\partial}{\partial t}\right)_R = \left(\frac{\partial}{\partial \tau}\right)_R \tag{17}$$

$$\left(\frac{\partial r}{\partial R}\right)_t = \left(\frac{\partial r}{\partial R}\right)_\tau + \left(\frac{\partial \tau}{\partial R}\right)_t\left(\frac{\partial r}{\partial \tau}\right)_R. \tag{18}$$

The resulting equation has an integrating factor $1 + L^2 r^{-2}$, giving the result:

$$\left(\frac{\partial \tau}{\partial R}\right)_t = -\frac{r^2 L}{L^2 + r^2}\left(\frac{\partial}{\partial R}\right)_\tau \left(\int^r \frac{L}{r^2(\partial r/\partial \tau)_R}\mathrm{d}r\right). \tag{19}$$

Again, this is an elliptic integral if $\Lambda = 0$ and hyperelliptic otherwise. It is closely related to the integral for $r(R, \tau)$. Evaluation of the integral, (19) includes a constant of integration, $g_0(R)$ say. This defines the origin of proper time, which may be different on different shells. This function is eliminated by choosing a common origin of time for all shells, i.e. at some time, say $t = 0$, we set $\tau = 0$ for all R. Knowing the distribution of matter at that time, $r(0, R)$, the equation $(\partial \tau/\partial R)_{t=0} = 0$ with $r = r(0, R)$ determines $g_0(R)$. This does not remove any generality from the problem, as the proper time is only defined locally for a shell. Adjusting where the time is measured from does not physically affect the spacetime. Using (15), (19) and the identity $\mathrm{e}^\nu \mathrm{d}t = \sqrt{1 + L^2 r^{-2}}\,(\mathrm{d}\tau - (\partial \tau/\partial R)_t \mathrm{d}R)$, the metric, (1) can be written

$$\begin{aligned}
\mathrm{d}s^2 = {} & \left(1 + \frac{L^2}{r^2}\right)\mathrm{d}\tau^2 - 2\left(1 + \frac{L^2}{r^2}\right)\left(\frac{\partial \tau}{\partial R}\right)_t \mathrm{d}\tau \mathrm{d}R - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \\
& - \frac{r^2 + L^2}{r^2(1 + 2E)}\left(\left(\frac{\partial r}{\partial R}\right)_\tau^2 + 2\left(\frac{\partial \tau}{\partial R}\right)_t \left(\frac{\partial r}{\partial R}\right)_\tau \left(\frac{\partial r}{\partial \tau}\right)_R \right. \\
& \left. + \left(\frac{\partial \tau}{\partial R}\right)_t^2 \left(\left(\frac{\partial r}{\partial \tau}\right)_R^2 - (1 + 2E)\right)\right)\mathrm{d}R^2.
\end{aligned} \tag{20}$$

All the functions in (20) are now known. $r(R, \tau)$ is given by inversion of (15), $(\partial r/\partial R)_\tau$ is given by (16) and $(\partial r/\partial \tau)_R$ by (14). The function $(\partial \tau/\partial R)_r$ is given by (19), noting

$$\left(\frac{\partial B(r, R)}{\partial R}\right)_\tau = \left(\frac{\partial B}{\partial R}\right)_r + \left(\frac{\partial r}{\partial R}\right)_\tau \left(\frac{\partial B}{\partial r}\right)_R. \tag{21}$$

The metric (20), with the equation of motion (14), the Misner-Sharp mass (12) and expression (19) for $(\partial \tau/\partial R)_t$ solves the problem in terms of the proper time and shell coordinate. The evolution of the system depends on the four free functions M, E, L and τ_0. One of these corresponds to a choice of the shell label, R. The others are fixed by the initial density, kinetic energy and angular momentum of the dust.

3.1. Shell Evolution

Equation (14) determines the motion of each individual shell. It may be written more clearly as:

$$\frac{1}{2}\left(\frac{\partial r}{\partial \tau}\right)_R^2 = E - V(r, R) = E - \left(\left(-\frac{GM}{r} - \frac{1}{6}\Lambda r^2\right)\left(1 + \frac{L^2}{r^2}\right) + \frac{L^2}{2r^2}\right). \tag{22}$$

Equation (22) is the familiar Newtonian equation except for the factor $(1 + L^2/r^2)$. This is a relativistic correction, which is due to the effective mass of the angular momentum. A physical evolution must have $E - V > 0$. The shape of the potential V determines which types of motion are possible, and the energy determines which regions of the potential a shell may penetrate. In the case $\Lambda = 0$, the potential was discussed by Bondi [6].

Figure 1a illustrates the possible shapes of $V(r, R)$ if $\Lambda = 0$, which is familiar as the potential for a Schwarzschild solution of mass M, angular momentum L and energy $\sqrt{1+2E}$. If Λ is non-zero, V contains the additional term $-\frac{1}{6}\Lambda r^2\,(1 + L^2/r^2)$. This has little effect at small r (only a constant shift $-\frac{1}{6}\Lambda L^2$), but dominates at large r. For positive Λ, it causes the potential to turn down towards $-\infty$, which can add an extra turning point in $r > 0$. For negative Λ, it causes the potential to turn up, and $V \to +\infty$. A single example of each sign is shown in figure 1b. The horizontal lines in figure 1b represent different choices of the energy E. The physical regions for a given value of E have $V < E$, indicated by the solid portions of the lines. The evolution of a shell depends on whether the energy intersects the potential above or below the turning points.

There are six possible types of evolution, which are labelled in figure 1b.

(i) **Expansion and recollapse.**
This occurs if $E - V > 0$ for $0 \leq r < R_{max}$ with $V(R_{max}) = E$. A shell in this region expands from a Big Bang at $r = 0$, to its maximum radius, R_{max}, and then recollapses.

(ii) **Bouncing Universe**
This occurs if $V(R_{min}) = E$ and $E - V > 0$ for $R_{min} < r < \infty$. This represents a shell which collapses from some initial size, reaches a minimum radius, R_{min}, and then expands to infinity.

(iii) **Expansion to/collapse from infinity.**
This occurs if $E - V > 0$ everywhere. A shell starting at infinity can collapse to the origin $r = 0$, or a shell created at $r = 0$ can expand to infinity.

(iv) **Oscillating Universe**
This occurs if $V(R_{min}) = E = V(R_{max})$ with $E - V > 0$ for $R_{min} < r < R_{max}$. A shell in this region oscillates between the two extremes, R_{min} and R_{max}.

(v) **Circular Orbits**
These occur when a shell sits at a minima (v.a) or maxima (v.b) of the potential. Orbits at maxima are unstable, those at minima stable.

(vi) **'Coasting' Universe**
A shell will coast if the potential has a repeated root at R_{circ}, but the shell is initially at $r \neq R_{circ}$. A shell starting at $R > R_{circ}$ will collapse, but the rate of collapse decreases as it gets closer to R_{circ}, so that it does not reach it in a finite time. A shell starting at $R < R_{circ}$ and initially expanding (perhaps from a big bang at $r = 0$) will coast outwards to the limiting value. The limits in each case are the circular orbits of (v).

If the energy in (vi) is adjusted to $E + \delta E$, the result is a 'hesitating' universe. This is one which expands, slows down ('hesitates') as it nears the turning point, before either undergoing accelerated expansion again ($\delta E > 0$) or recollapsing ($\delta E < 0$) (see [18] for a description of this in the FRW context).

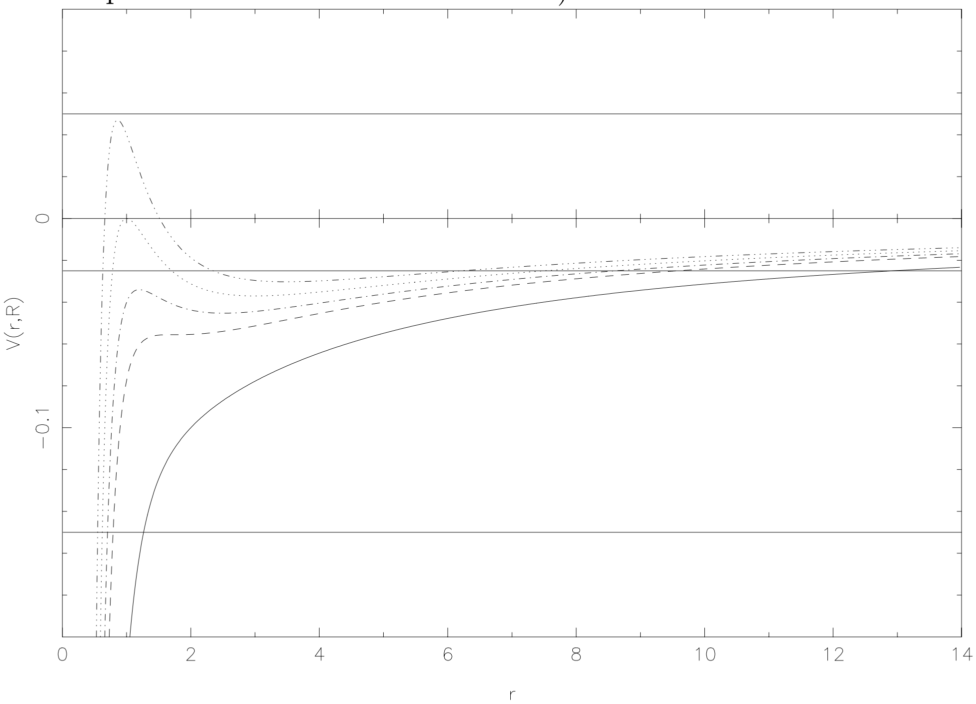

Figure 1a Possible potential shapes for $\Lambda = 0$. As the parameter $k = L/GM$ changes, the potential changes (starting with the lowest, solid curve) from having no roots and no turning points ($k < 2\sqrt{3}$); no roots and a point of inflection ($k = 2\sqrt{3}$); two turning points and no roots ($2\sqrt{3} < k < 4$); one repeated root ($k = 4$) and finally two roots ($k > 4$).

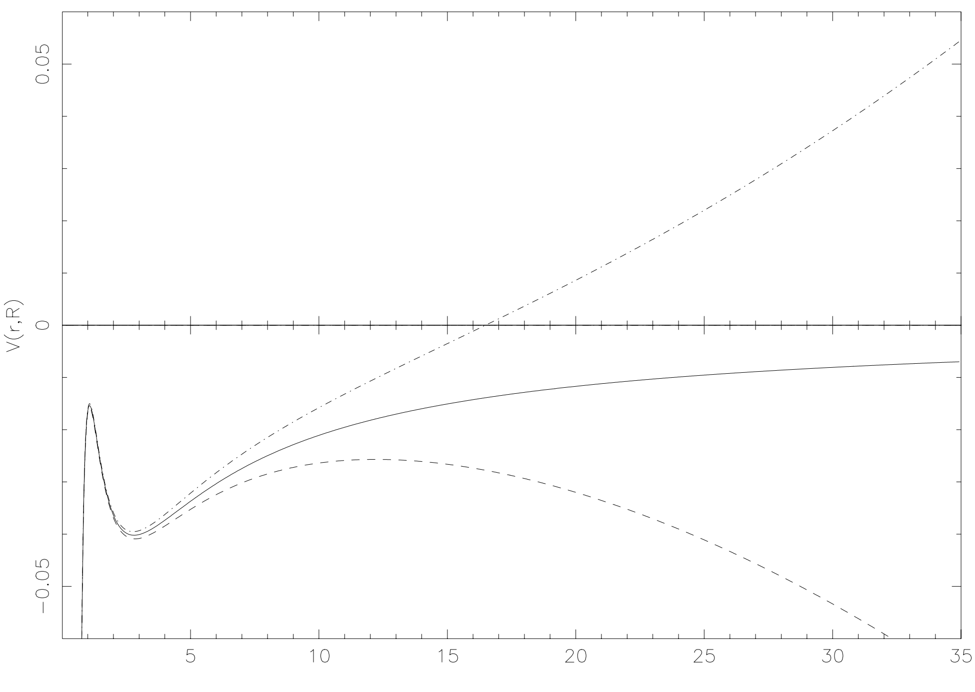

Figure 1b Shape of potential for $\Lambda > 0$ (solid line), $\Lambda = 0$ (dashed line) and $\Lambda < 0$ (dot-dash line). The horizontal lines correspond to different choices of the energy E. Physical regions have $E > V$, and are indicated by the solid parts of these lines for the $\Lambda = 0$ potential.

3.2. Global Behaviour

The universe is composed of an infinite number of shells of the type discussed in the previous section. These shells lie within one another, and so care has to be taken when constructing a global solution. In general, a physically reasonable solution is one which starts from reasonable initial data - either a Big Bang type singularity or a regular matter distribution, and evolves without the formation of shell crossing singularities.

Behaviour of the Functions of R There are four free functions of R in the solution. The time function $\tau_0(R)$ specifies the initial distribution of matter. In big bang models, $r = 0$ for all R at $t = 0$, which we take to be $\tau = 0$ on every shell. In collapse models, it is sensible to choose $r = R$ at $t = 0$. The mass function M obeys equation (12). If the spacetime is to satisfy the energy conditions, we require $\epsilon > 0$ and so M' and r' must have the same sign. Finally, in studies of gravitational collapse, it is conventional to require the initial data to be regular and all functions to be C_∞ until a singularity forms. This leads to the conditions $M(R) = M_3R^3 + M_5R^5 + ...$, $E(R) = E_2R^2 + E_4R^4 + ...$ and $L^2(R) = L_4R^4 + L_6R^6 + ...$ (see [12]). In the other models, for instance the expansion and recollapse case, the condition of C_∞ evolution is not necessarily relevant, as the universe is born from a singularity.

Curvature Singularities A curvature singularity occurs when an invariant diverges, for instance R^α_α. Expressions (2) and (12) give $R^\alpha_\alpha = -\left(8\pi G T^\alpha_\alpha/c^4 + 4\Lambda\right) = -\left(16\pi G^2 M'/(c^4(L^2 + r^2)r') + 4\Lambda\right)$. This is singular if $r'/M' = 0$, which corresponds to shell crossing, or when $r = 0$ provided $L = 0$ there. In [12], they took $L \propto R^4$, in which case the centre can become singular when the central shell $R = 0$ reaches it. As further shells reach $r = 0$, the point remains singular since there is a non-zero point mass there. In general, the central singularity can be naked or clothed (i.e. a black hole). It can be naked only for the time between the shell $R = 0$ reaching $r = 0$ and any subsequent shell arriving. Photons cannot escape from any region with $r < 2GM(R)$. For the central shell, which has $M = 0$, photons may escape from the origin, but this will not be the case for shells with $M \neq 0$. This was also mentioned by Harada et al. The formation of naked or clothed singularities will be discussed for some specific examples later.

Coordinate Singularity at $E = -1/2$ It is clear from the metric (20) that if $E = -\frac{1}{2}$, the g_{RR} coefficient is singular, unless r' is also zero at that point. This is a coordinate singularity and may be removed by a coordinate transformation $R \to R'$ with $\mathrm{d}R'/\mathrm{d}R = 1/\sqrt{1 + 2E}$ near the singularity. In the new coordinates, $(\partial r/\partial R')_t = 0$ at this point, but this does not represent shell crossing as $\mathrm{d}M/\mathrm{d}R' = 0$ also. This is the equator of the universe. On the other side of the singularity, the universe is in reverse, with $M' < 0$ and $r' < 0$. The other half of the universe may be a mirror image of the first, or of any other model which has the same values of M, L and E at the equator. In this way, two half universes are joined together at the coordinate singularity. This is similar to the $k = -1$ FRW model (33). The point $R = 1$ is singular, but writing $R = \sin\chi$, this singularity is removed. The universe consists of two identical hemispheres, joined at the coordinate singularity $R = 1$.

Shell Crossing The solution breaks down when a shell crossing takes place, which is indicated by $r' = 0$ and the density, ϵ, diverging. A similar problem occurs in the Tolman-Bondi solution (with $L = 0$), and has been discussed in detail in [19]. In more general solutions of relativistic stellar dynamics (e.g. [20], [21]), such problems do not arise. These models use the Einstein-Vlasov equations, the evolution is followed in phase space and streams may cross. However, the non-static Einstein-Vlasov system is difficult to solve. In fact, this model could be regarded as Einstein-Vlasov, because the dust is assumed to follow geodesic paths. The phase space density is proportional to $\delta(p_r - \dot{r}(r,t))$ at each point. There is no radial velocity dispersion and hence no radial pressure, but the phase space density is divergent, since the momentum is confined to lie in a plane at each point. In more general Einstein-Vlasov systems, there will be some radial velocity dispersion. This gives rise to radial pressure, which has been ignored in this model and is likely to be important at shell crossing. In fact, the problem at shell crossing is not with gravity, but with the solution. The shell crossing problem could be avoided by looking at the evolution in phase space, and indeed the divergent density is only a surface layer, which can be treated in general relativity ([22]). Our method of solution, however, relies on the fact that the shells are distinct, which gives an equation of motion for each shell that is independent of the others. Moreover, when the density becomes large, interactions between the dust particles are likely to become important. Thus, this model shall only be regarded as valid until a shell crossing first takes place.

A shell crossing is indicated by $(\partial r/\partial R)_t = 0$. This derivative is taken at constant t rather than by holding one of the other times, τ or T, constant. That this is the correct derivative is clear from (12), which tells us that $\epsilon \propto (\partial r/\partial R)_t^{-1}$. When $r' = 0$, the density diverges as the shells attempt to occupy the same space. This is not true if $M' = r' = 0$, which may occur at the equator of a closed universe, as discussed previously, or where $\epsilon = 0$. The latter indicates a region in which the shells are unoccupied and hence shell crossing is not a problem. With this exception, the condition for no shell crossing is that

$$\left(\frac{\partial r}{\partial R}\right)_t = \left(\frac{\partial r}{\partial R}\right)_\tau + \left(\frac{\partial \tau}{\partial R}\right)_t \left(\frac{\partial r}{\partial \tau}\right)_R \neq 0 \quad \forall\ \tau,\ R. \tag{23}$$

The functions $(\partial r/\partial \tau)_R$ and $(\partial \tau/\partial R)_t$ are given by (14) and (19) respectively. $(\partial r/\partial R)_\tau$ may be obtained from (21) or by differentiating (14) and then evaluating the expression:

$$\left(\frac{\partial r}{\partial R}\right)_\tau = \sqrt{2(E-V)} \int_{r(0,R)}^{r} \frac{GM'r^2 + (GML^2)' - LL'r + E'r^3 + \frac{1}{3}\Lambda LL'r^3}{r^3(2(E-V))^{\frac{3}{2}}} \mathrm{d}r. \tag{24}$$

Certain choices for M, E, L and τ_0 will give solutions in which shell crossing never occurs. Some necessary conditions for such an evolution will now be discussed. In the following, $M' > 0$ is assumed, and so physically $r' > 0$ is required. Regions with both of these negative are equivalent to this case by relabeling the shell coordinate R.

If a solution begins with a big bang at $t = 0$, then equation (14) tells us that for $r \ll r_1$ (where $r_1(R)$ is the location of the first turning point of $V(r,R)$), $r \sim (5\tau)^{2/5}(GML^2/2)^{1/5}$. Integration of (19) with the boundary condition $\tau = 0 = t$ gives $(\partial\tau/\partial R)_t = -5L\mu^4\,(L/\mu^4)'\,\tau/(L^2+\mu^4\tau^{\frac{4}{5}})$ where $\mu(R) = (25GML^2/2)^{1/10}$. Using these in (23), the condition for evolution without shell crossing becomes $(\ln(\mu^5/L))' > 0$, but this just states $M' > 0$, which is assumed.

In unbounded evolution $(\partial r/\partial\tau)_R^2 \sim \frac{1}{3}\Lambda r^2 + 2E + \frac{1}{3}\Lambda L^2$ asymptotically. If $\Lambda \neq 0$, this is the same for all shells. If $\Lambda = 0$ equation (23) becomes $(\partial r/\partial R)_t \sim A + \frac{E'}{E}r$ for large r. If $E' < 0$, shell crossing will inevitably occur. We therefore require $E' \geq 0$ for unbounded expanding evolutions. For a collapsing evolution that begins at infinity, the sign of the E'/E term changes and the reverse must be true, $E' \leq 0$.

If any shell has a maximum radius, all the shells within it must. Similarly, if a shell has a minimum radius, all those exterior to it must. Moreover, the maximum or minimum radius must be an increasing function of R. This applies to the maximum radius in the expansion/recollapse case, the bouncing point for a bouncing region and the limiting radius in a coasting evolution.

In a bouncing region of the spacetime, the bounce point, i.e. the part of the universe that is bouncing, must move outwards 'supersonically', that is faster than the surrounding shells expand. In a dust collapse model, it has been shown for the Lemaître-Tolman-Bondi case by Hellaby and Lake [19] among others that it is always possible to choose the initial conditions in such a way as to ensure there is no shell crossing. The freedom in τ_0 allows the same here, although equally initial conditions can be chosen which ensure shell crossing *does* occur.

Finally, for a coasting region of the universe (case (vi)), $(\partial r/\partial\tau)_R^2 \sim f(R)(r_0(R)-r)$ for every shell asymptotically. Equations (14), (19) and (23) tell us that $(\partial r/\partial R)_\tau \sim r_0' + \tau(f'(R) - L^2 f(R)(\ln(L/r_0^2))'/(L^2+r_0^2))\exp(-f(R)\tau)$. If $\mathrm{d}r_0/\mathrm{d}R$ and the bracketed term are both positive, there will be no shell crossing asymptotically. This does not exclude shell crossing at earlier times, however.

A family of self-similar solutions to this model can be found. In that case, the shell crossing conditions can be written as inequalities for the free constants in the solution. The self-similar models will be discussed in a separate paper. All the types of evolution (i)-(vi) can occur for the whole universe without shell crossing in the self-similar case, except for (iv), the oscillating universe.

In the oscillating case, shells will be continually changing directions for all time and so shell crossing seems quite likely. In the next section a solution undergoing small oscillations will be derived by perturbing an Einstein Cluster. If a solution undergoing large oscillations is to exist, a necessary condition is that the period of oscillations in t is the same for all shells. This is difficult to evaluate in general, as an explicit expression for e^ν is not known. Progress can be made using (23). If the system is undergoing oscillations, then necessarily $r(R, \tau + n\Pi(R)) = r(R, \tau)$, $\forall n \in Z$, i.e. each shell oscillates with some period $\Pi(R)$ in τ. If we denote the limiting radii of the oscillations of each shell by $r_{in}(R)$ and $r_{out}(R)$, then this period is given by the integral:

$$\Pi(R) = 2\int_{r_{in}(R)}^{r_{out}(R)} \frac{\sqrt{3}r^2}{\sqrt{\Lambda r^6 + (6E + \Lambda L^2)r^4 + 6GMr^3 - 3L^2r^2 + 6GML^2r}}\mathrm{d}r. \tag{25}$$

From (23) and (19), it is possible to relate the shell crossing condition after n periods have passed to the values of the functions in the first period:

$$\begin{aligned}\left(\left(\frac{\partial r}{\partial R}\right)_t\right)_{\tau=\tau_0+n\Pi(R)} &= \left(\frac{\partial r}{\partial R}\right)_{\tau_0} - \frac{Lr^2}{L^2+r^2}\left(\frac{\partial r}{\partial \tau}\right)_{\tau_0}\left(\frac{\partial}{\partial R}\right)_{\tau_0}\left(\int_0^{\tau_0}\frac{L}{r^2}\mathrm{d}\tau\right) \\ &+ n\left(\frac{\partial r}{\partial \tau}\right)_{\tau_0}\frac{r^2}{L^2+r^2}\left[-\Pi'(R) - 2L\frac{\mathrm{d}}{\mathrm{d}R}\left(\int_0^{\Pi(R)/2}\frac{L}{r^2}\mathrm{d}\tau\right)\right]. \end{aligned} \tag{26}$$

This uses the fact that $(\partial/\partial R)_{\tau_0+n\Pi(R)} = (\partial/\partial R)_{\tau_0} - n\Pi'(R)(\partial/\partial\tau)_{\tau_0}$. Taking $0 \leq \tau_0 \leq \Pi(R)$, the first line of (26) is the value of $(\partial r/\partial R)_t$ on the first cycle. The second line is proportional to $(\partial r/\partial\tau)_{R,\tau_0}$, which changes sign during a single oscillation and n, which may be arbitrarily large. Thus, even if shell crossing does not occur during the first period, it will occur inevitably for large enough n, unless the term in square brackets is 0. This is a function of R only and gives the condition:

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[\int_{r_{in}(R)}^{r_{out}(R)}\frac{1}{\sqrt{2(E-V(r,R))}}\mathrm{d}r\right] + L\frac{\mathrm{d}}{\mathrm{d}R}\left[\int_{r_{in}(R)}^{r_{out}(R)}\frac{L}{r^2\sqrt{2(E-V(r,R))}}\mathrm{d}r\right] = 0. \tag{27}$$

This is the statement that the period in t of every shell is the same. It is a necessary condition on the functions M, L and E for the evolution to proceed without shell crossing. The condition is only sufficient if there is no crossing during the first oscillation. The additional freedom in the initial data allows the construction of oscillating universes which have no shell crossing on the first cycle and hence never.

3.3. Einstein Clusters

If every shell is in a circular orbit the spacetime is static. In a circular orbit $\dot{r} = \ddot{r} = 0$, i.e. $V(r,R) = E(R)$ and $\partial V/\partial r = 0$. Labeling the shells by $R = r$ this gives

$$\frac{L^2}{R^2} = \frac{\frac{GM}{R} - \frac{1}{3}\Lambda R^2}{1 - 3\frac{GM}{R}}, \qquad 2E = -1 + \frac{(1 - \frac{1}{3}\Lambda R^2 - 2\frac{GM}{R})^2}{1 - 3\frac{GM}{R}}. \tag{28}$$

A circular orbit is stable if

$$\left(\frac{\partial^2 V}{\partial r^2}\right)_{|r=R} = \frac{\tilde{M}(1-6\tilde{M}) + \Lambda R^2(6\tilde{M} - \frac{5}{3})}{R^2(1-3\tilde{M})} > 0. \tag{29}$$

For this static case, the integrand in (19) is independent of τ and so $(\partial\tau/\partial R)_t$ is given easily. Alternatively, equation (9) gives

$$\nu = \int \frac{\tilde{M} - \frac{1}{3}\Lambda R^2}{R(1 - 2\tilde{M} - \frac{1}{3}\Lambda R^2)}\mathrm{d}R. \tag{30}$$

The solution in (τ, R) coordinates is given explicitly by specifying $\tilde{M}$ and Λ and in (t, R) coordinates by specifying ν and Λ. However, quadratures are required to determine $\tilde{M}$ or ν from the matter distribution ϵ.

Zero Cosmological Constant For $\Lambda = 0$, this solution is the Einstein Cluster [1]. Expression (30) and the stability criterion (29), $0 \leq \tilde{M} \leq \frac{1}{6}$, agree with those derived in a more roundabout way in [2]. Expression (28) requires $\tilde{M} \leq \frac{1}{3}$ if L is to be real. The case $\tilde{M} = \frac{1}{3}$ corresponds to a universe composed of radiation rather than dust and will be discussed elsewhere. This spacetime has $-\frac{1}{18} \leq E$, which means there is never enough mass to close the universe (i.e. E can never equal $-1/2$).

Non Zero Cosmological Constant If $\Lambda > 0$, but $L = 0$, the solution is the Einstein Static Universe, which has $\tilde{M} = \frac{1}{3}\Lambda R^2$ everywhere. The energy $E = -\frac{3}{2}\tilde{M} = -\frac{1}{2}$ when $\tilde{M} = \frac{1}{3}$, so the universe is closed with a maximum radius $R_{max} = 1/\sqrt{\Lambda}$. From (29), this universe is unstable everywhere. If both Λ and L are non zero, L^2 changes sign when $\tilde{M} = \frac{1}{3}$ and when $\tilde{M} = \frac{1}{3}\Lambda R^2$. One of these equations always gives a maximum radius for the cluster. This contrasts with the $\Lambda = 0$ case in which infinite clusters are possible. If $\tilde{M} = \frac{1}{3} = \frac{1}{3}\Lambda R^2$ at the same point, L^2/R^2 will be finite there, and the energy E reaches $-\frac{1}{2}$, the energy required to close the universe. The static model can therefore represent either a cluster or a closed static universe. In the latter, the maximum radius of the universe, $R_{max} = 1/\sqrt{\Lambda}$, is independent of the L distribution.

From (29), the universe is stable if $x - \frac{1}{12}\sqrt{y} < \tilde{M} < x + \frac{1}{12}\sqrt{y}$, where $x = \frac{1}{12} + \frac{1}{2}\Lambda R^2$ and $y = 1 - 28\Lambda R^2 + 36\Lambda^2 R^4$. This is satisfied near $R = 0$ if $\frac{5}{3}\Lambda R^2 < \tilde{M}$, but it never holds for $\tilde{M} > \frac{1}{6}$. The outer regions of a closed static universe must always be unstable, which is the case everywhere in the Einstein Static Universe.

The $\Lambda < 0$ case is very similar to $\Lambda = 0$. Only cluster models are possible, which may be stable or unstable, and $\tilde{M} < 1/3$ so that L is real. The stability criterion is slightly relaxed. The cluster is still stable where $0 < \tilde{M} < 1/6$ and is unstable for $\tilde{M} > 5/18$. For $1/6 < \tilde{M} < 5/18$, the cluster is stable if $R > \sqrt{(\tilde{M}(6\tilde{M} - 1))/(|\Lambda|(5/3 - 6\tilde{M}))}$ and unstable otherwise.

Cosmic Censorship The Einstein clusters are the asymptotic states for 'coasting' evolution. Such evolutions have the same E, M and L, but the evolution begins away from the circular orbit. In the $\Lambda = 0$ case, marginally bound coasting collapse was discussed in [12]. This solution has $E = 0$ and hence $L = 4GM$ (from (28)). The solution exhibits a globally naked singularity at the origin, which is partially due to the presence of a naked singularity in the asymptotic Einstein cluster. For an Einstein cluster with a singularity at the centre $\epsilon \sim R^{-\delta}$ and $\tilde{M} \propto R^{2-\delta}$ near $R = 0$ with $0 < \delta \leq 2$ (so that $\tilde{M}$ is bounded). The expressions (10) and (30) allow the radial photon equation $\mathrm{d}R/\mathrm{d}t = \exp(\nu - \lambda)$ to be evaluated. For $0 < \delta < 2$, we find $\exp(\nu - \lambda) \sim (1 - 2\tilde{M}_0 R^{2-\delta})^{(\delta-1)/2}$ near $R = 0$, where $\tilde{M}_0 = \lim_{R\to 0}(\tilde{M}/R^{2-\delta})$ is a constant. In the case $\delta = 2$ this becomes $\exp(\nu - \lambda) \sim R^{2\tilde{M}_0/(1-2\tilde{M}_0)}$. In either case, the photon equation is integrable, photons can propagate from the origin in finite time and the singularity is naked. This excludes the marginal case $\delta = 2$ and $\tilde{M}_0 = \frac{1}{3}$. This is the radiation limit, and will be discussed elsewhere.

All these Einstein clusters exhibit locally naked singularities, which may be globally naked depending on the structure of the rest of the spacetime. However, if the Einstein cluster is to be formed as the endstate of coasting collapse, the orbit of each shell in the spacetime must be located at a maximum of the potential, i.e. at an unstable point, so that $E - V > 0$ in a region about that point. If $\Lambda = 0$, this requires $\tilde{M} > \frac{1}{6}$. This rules out all the cases with $0 < \delta < 2$, which have $\tilde{M} \to 0$ as $R \to 0$. Only if $\delta = 2$ and $\tilde{M}_0 > \frac{1}{6}$ can the cluster form in collapse. This leaves a one parameter family of collapse solutions which do have naked singularities in the endstate, characterized by the value of $\tilde{M}$ in that endstate. The case considered by Harada *et al* [12] is one such example, which has $\tilde{M}_0 = \frac{1}{4}$. The presence of a non-zero Λ does not affect the solution near $r = 0$. Again, a locally naked singularity will be present if $\tilde{M} \sim \tilde{M}_0 \geq \frac{1}{6}$, a constant, near $R = 0$. This may be globally naked, depending on the structure of the spacetime away from $R = 0$.

It is possible in this way to produce naked singularities readily in this model. However, fine tuning of the functions E, L and M is required, and if the solutions form in collapse, they must be at maxima of the potential. For that reason, the solutions are unstable, since it is likely that even spherically symmetric perturbations will cause the cluster to evolve away from the static solution, resulting in rapid clothing of the singularity.

Oscillating Einstein Cluster Perturbation of a stable Einstein cluster produces a solution undergoing oscillations. Keeping M and L unchanged, but letting $r \to R + \delta r(R,t)$, $\nu \to \nu_0(R) + \delta\nu(R,t)$ and $E \to E_0(R) + \delta E(R)$, we obtain from (14)

$$\frac{\partial^2 \delta r}{\partial \tau^2} = -\frac{1}{2}\frac{\partial^2 V}{\partial r^2}_{|r=R}\delta r = -\frac{\tilde{M}(1-6\tilde{M}) + \Lambda R^2(6\tilde{M} - \frac{5}{3})}{R^2(1-3\tilde{M})}\delta r. \qquad (31)$$

$(\partial^2 V/\partial r^2) > 0$ necessarily, as the cluster being perturbed is stable. This is the simple harmonic motion equation - each shell undergoes SHM with period $\omega = \sqrt{(\tilde{M}(1-6\tilde{M}) + \Lambda R^2(6\tilde{M} - \frac{5}{3}))/(R^2(1-3\tilde{M}))}$ in τ. To avoid shell crossing, the period in t must be the same for each shell. To lowest order, the period in t is $\Omega = \omega e^{\nu_0}(1 + L^2/R^2)^{-\frac{1}{2}}$. Requiring this to be constant we have

$$\tilde{M}'\left(\frac{12\tilde{M}^2 - 12\tilde{M} + 1 + \Lambda R^2(\frac{7}{3} + 4\tilde{M}) + 2\Lambda^2 R^4}{2\tilde{M}(1 - 6\tilde{M} + \frac{4}{3}\Lambda R^2)(1-3\tilde{M})}\right) = \frac{1}{R}. \qquad (32)$$

In the case $\Lambda = 0$, this is integrable and $R = (\sqrt{\tilde{M}}(1-2\tilde{M})^{2/3})/(\Omega(1-3\tilde{M})^{5/6})$. This reaches a maximum at $\tilde{M}_{max} = (1 - \sqrt{2/3})/2$, which means there are two solutions, one with $0 < \tilde{M} < \tilde{M}_{max}$ and one with $\tilde{M}_{max} < \tilde{M} < 1/6$. The functions $R(\tilde{M})$ and

$M(R)$ for the two solutions are shown in figure 2. For the second solution, $\mathrm{d}M/\mathrm{d}R = 0$ before R_{max} is reached. The solution must be cut off there, to ensure $\epsilon > 0$. It can be matched smoothly onto an exterior, since the density reaches 0 at the boundary.

This solution is approximate, since terms of order δ^2 and higher have been ignored. These terms will be important at late times, which will lead to shell crossing eventually. The full condition (27) can be used to compute the $\delta E(R)$ necessary to avoid shell crossing completely.

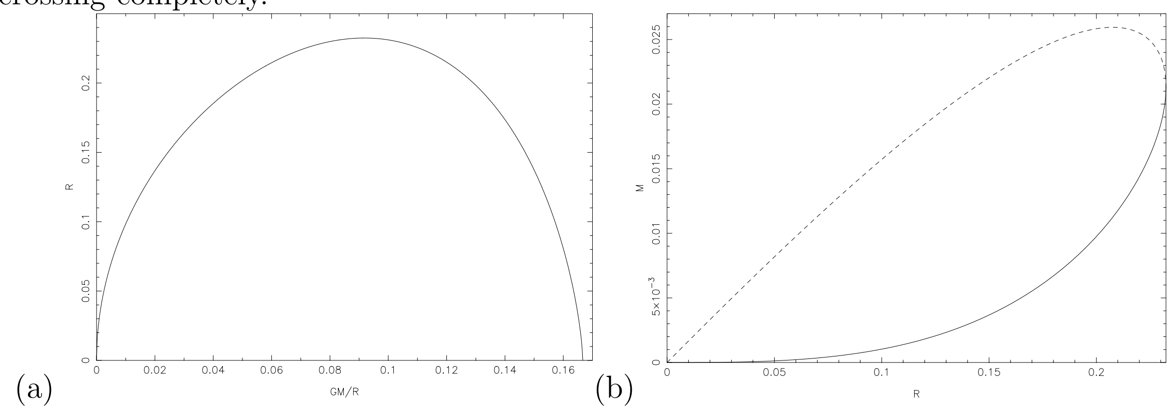

Figure 2 (a) Plot of $R(\tilde{M})$ for the $\Lambda = 0$ vibrating Einstein Cluster. The curve has a maximum at $\tilde{M} = \tilde{M}_{max} = \frac{1}{2}\left(1 - \sqrt{\frac{2}{3}}\right)$. (b) M(R) for the two branches of (a). The solid line is the branch with $0 < \tilde{M} < \tilde{M}_{max}$ and the dashed line is the branch with $\tilde{M}_{max} < \tilde{M} < \frac{1}{6}$.

3.4. Matching

This model may be used to represent a region within a background spacetime. In this context, it is useful to know how to match this solution onto other metrics.

FRW If we take $M = \rho_0 R^3$, $L = 0$, $2E = kR^2$ and $r = Ra(t)$, then this solution is the FRW metric with scale factor $a(t)$:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{1}{1+kR^2}\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right). \quad (33)$$

This may be open ($k > 0$), flat ($k = 0$) or closed ($k < 0$). If the functions M, E and L are chosen to have this form asymptotically, the solution matches smoothly onto an FRW exterior. Such a model represents an FRW universe containing an anisotropic pocket which may recollapse to form a black hole. Such a 'swiss-cheese' universe, having Tolman-Bondi regions within an FRW exterior has been considered previously, for instance in [23]. In this model, an example of a one-cell universe is to choose $2GM = R^3$, $2E = kR^2$ and $L = R^2(R-1)^2$ for $R \leq 1$ and $L = 0$ for $R \geq 1$. This solution does not exhibit shell crossing for $k = \pm 0.25$ and $\Lambda = -1.0$, 0.0 or 1.0. In each case, the universe is born in a big bang and the core recollapses to form a black hole. The evolution of the rest of the universe is determined by k and Λ, and it either expands forever, or recollapses in a big crunch.

Schwarzschild-de Sitter If the matter distribution is finite in extent, then spherical symmetry requires the exterior vacuum metric to be Schwarzschild-de Sitter:

$$\mathrm{d}s^2 = F(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{F(r)} - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{34}$$

In this, $F(r) = 1 - 2Gm/r - \frac{1}{3}\Lambda r^2$. The matching of this solution onto a Schwarzschild-de Sitter exterior was discussed partially in [6]. We use the Israel-Darmois [22] matching conditions for the match. If the density is zero at the edge of the interior region, $M' = 0 = L$ at the boundary. The match can be made smoothly by taking $M = constant$ and $L = 0$ for the whole exterior region. If the density is non-zero at the boundary, there is only a jump discontinuity in the energy tensor T^μ_ν there. This is classified as a boundary surface, over which the first (g_{ij}, the induced metric) and second (K_{ij}, the extrinsic curvature) fundamental forms are continuous. We denote the interior coordinates as $R, \tau, r(R,\tau), \theta, \phi$ and the exterior coordinates by $t, \tilde{r}, \tilde{\theta}, \tilde{\phi}$ and take the boundary to be at $R = R_0$, $\tilde{r} = \tilde{r}_0(t)$. Comparing the metrics induced on the boundary by the interior solution (20) and the exterior solution (34), and choosing the angular coordinates θ and ϕ to be continuous, continuity of the first fundamental form requires that:

$$\tilde{r}_0(t) = r(R_0, \tau) \tag{35}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{\frac{1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2 - \frac{\frac{\mathrm{d}r}{\mathrm{d}t}^2}{1 - \frac{2GM}{r} - \frac{1}{3}\Lambda r^2}}{1 + \frac{L^2}{r^2}}}. \tag{36}$$

In (36), $r = \tilde{r}_0(t) = r(\tau, R_0)$. Continuity of the second fundamental form ensures that because the particles in the shell undergo geodesic motion of the interior metric, their motion will be geodesic of the exterior as well. Geodesics of the metric (34) are characterized by two conserved quantities, the energy $\tilde{E}$ and angular momentum $\tilde{L}$, which are the same for every particle on the boundary by spherical symmetry. The geodesic equations then give an expression for the evolution of the boundary. A second expression is derived from the interior solution, since $\mathrm{d}r(\tau, R_0)/\mathrm{d}t = (\mathrm{d}\tau/\mathrm{d}t)(\partial r/\partial\tau)_{R=R_0}$ and these are given by (36) and (22). Equating the two expressions for $(\mathrm{d}\tilde{r}_0/\mathrm{d}t)^2/F^2(\tilde{r}_0)$, we find

$$1 - \frac{1}{\tilde{E}^2}\left(1 + \frac{\tilde{L}^2}{\tilde{r}_0^2}\right)F(\tilde{r}_0) = \frac{-\left(1 - \frac{2GM(R_0)}{\tilde{r}_0} - \frac{1}{3}\Lambda\tilde{r}_0^2\right)\left(1 + \frac{L^2(R_0)}{\tilde{r}_0^2}\right) + 2E(R_0) + 1}{\frac{2G}{\tilde{r}_0}\left(M(R_0) - m\right)\left(1 + \frac{L^2(R_0)}{\tilde{r}_0^2}\right) + 1 + 2E(R_0)}. \tag{37}$$

It is clear that the two sides of equation (37) are consistent only if $\tilde{L} = L(R_0)$, $m = M(R_0)$ and $\tilde{E}^2 = 1 + 2E(R_0)$. With this identification, the evolution of the boundary is given by (37) in both the interior and exterior coordinates. The two regions are connected by a common value of the boundary areal radius.

This matching also applies to a three zone model. Suppose that there is a region with angular momentum for $0 \leq r \leq \tilde{r}_{in}(t)$, with $R = R_{in}$ and $r(R_{in}, \tau) = \tilde{r}_{in}(t)$ at the outer boundary; then a vacuum (Schwarzschild-de Sitter) region for $\tilde{r}_{in} \leq r \leq \tilde{r}_{out}$ and for $\tilde{r}_{out} \leq r$ an FRW region, with $\rho_0 = M(R_{in})/R_{out}^3$ and $R = R_{out}$ at the inner boundary. The two boundaries evolve as geodesics in the Schwarzschild region, with $m = M(R_{in})$, $\tilde{L}_{in} = L(R_{in})$, $\tilde{E}_{in}^2 = 1 + 2E(R_{in})$, $\tilde{L}_{out} = 0$ and $\tilde{E}_{out}^2 = 1 + kR_{out}^2$. The matching is achieved by following the evolution of the boundary in each set of coordinates, and relating common values of the boundary radius. Such a model is applicable to a universe with two big bangs, where the angular momentum region is born within a void in the expanding background some time after the main big bang.

4. Solutions

4.1. Solution as Elliptic Integrals for $\Lambda = 0$

If $\Lambda = 0$, the polynomial inside the square root in the integrals (15) and (19) is a quartic. The solutions can then be written in terms of elliptic integrals, according to the definition of [24]. The integrals are respectively:

$$I_1 = \int_{r_0}^{r} \frac{r^2}{\sqrt{2Er^4 + 2GMr^3 - L^2r^2 + 2GML^2r}} \mathrm{d}r \tag{38}$$

$$I_2 = \int_{r_0}^{r} \frac{1}{\sqrt{2Er^4 + 2GMr^3 - L^2r^2 + 2GML^2r}} \mathrm{d}r. \tag{39}$$

The value of these integrals depends on the location of the roots of the polynomial. If $E = 0$, it can be shown that:

$$I_1 = \frac{2}{3\sqrt{2GM}} \sqrt{r\left(r^2 - \frac{L^2}{2GM}r + L^2\right)} + \frac{\sqrt{2}L^2}{3(GM)^{\frac{3}{2}}} J_1(r; r_0; 0, r_-, r_+) - \frac{L^2}{\sqrt{2GM}} J_0(r; r_0; 0, r_-, r_+) \tag{40}$$

$$I_2 = \frac{1}{\sqrt{2GM}} J_0(r; r_0; 0, r_-, r_+). \tag{41}$$

In this, $r_\pm = L^2(1 \pm \sqrt{1 - 16G^2M^2/L^2})/4GM$, and the functions J_n are given by:

$$J_n(r; r_0; a, b, c) = \int_{r_0}^{r} \frac{x^n \mathrm{d}x}{\sqrt{|(x-a)(x-b)(x-c)|}} \tag{42}$$

This result is obtained by differentiation of $\sqrt{r(r^2 - L^2r/2GM + L^2)}$ with respect to r and integration of the result. If $E \neq 0$ and the roots of $r^4(E - V(r, R))$ are written as 0, r_1, r_2 and r_3, the integrals become:

$$I_1 = \frac{r\sqrt{E - V(r, R)}}{\sqrt{2}E} + \frac{1}{2E}\sqrt{\frac{GM}{2L^2}} J_{-1}\left(\frac{1}{r}; \frac{1}{r_0}; \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}\right) - \frac{\sqrt{GML^2}}{2\sqrt{2}E} J_1\left(\frac{1}{r}; \frac{1}{r_0}; \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}\right) \tag{43}$$

$$I_2 = -\frac{1}{\sqrt{2GML^2}} J_0\left(\frac{1}{r}; \frac{1}{r_0}; \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}\right) \tag{44}$$

These are derived by considering $\mathrm{d}(r\sqrt{E-V(r,R)})/\mathrm{d}r$, integrating the result and changing variable to $u=r^{-1}$ in the resulting integrals. The roots r_i can be obtained in the standard way for cubic equations (see for instance [24]). The value of the eliptic integral J_n depends on the number of real roots. If there are 3 roots (which may be repeated), the integrals are given in [25]. If there is only one real root, the term in the square root in J_n may be written in the form $(x-a)((x-d)^2+f^2)$. In this case, writing $x=a+p\tan^2(\phi/2)$ with $p=\sqrt{(a-d)^2+f^2}$, the integrals reduce to a combination of elliptic integrals and elementary functions.

4.2. Potentials with Repeated Roots

An interesting example for non zero Λ is the case in which $E(R)-V(r,R)$ has a triple root in r. This ensures $\partial V/\partial r$ has a repeated root, which gives the condition $\Lambda Y = Q_\pm(X)$ for $X=L^2/(3G^2M^2)$ and $Y=3L^2$. The repeated root is located at $r=r_0=2\sqrt{XY}P_\pm(X)/3$. This result and the functions $Q_\pm(X)$ and $P_\pm(X)$ are given in the appendix. The condition $\Lambda Y=Q_\pm(X)$ can be rewritten as $Q_\pm(X)/X=9G^2M^2\Lambda$. Given a value for Λ and $M(R)$, this determines L^2. For $\Lambda>0$, this has solutions in $X\geq 3.75$ for Q_+ and in $3.75\leq X\leq 4.0$ for Q_-. For $\Lambda<0$, this has solutions only for $Q_-(X)$ and $X\geq 4.0$. We shall concentrate on the former case, $\Lambda>0$. In that case the functions $Q_\pm(X)/X$ have a maximum at $X=3.75$, which constrains $GM\leq 2/(75\sqrt{\Lambda})=GM_{max}$. This suggests taking $z=M/M_{max}=z(R)$ as the shell label. X is then given by the relation $Q_\pm(X)/X=4/(625z^2)$, and so $X=X(z)$ only. Requiring r_0 to be a root of $E-V$ fixes $E=V(r_0,z)$. The functions X, L, r_0 and E are then given in terms of Λ and z by the expressions

$$\frac{Q_\pm(X)}{X}=\frac{4}{625}z^2\rightarrow X(z)\text{ implicitly} \tag{45}$$

$$L^2=\frac{1}{\Lambda}\tilde{L}^2=\frac{1}{\Lambda}\frac{4}{1875}z^2X(z) \tag{46}$$

$$r_0=\frac{1}{\sqrt{\Lambda}}\tilde{r}_0(z)=\frac{1}{\sqrt{\Lambda}}\frac{4}{75}zX(z)P_\pm(X(z)) \tag{47}$$

$$E(z)=-\left(\frac{1}{6}\tilde{L}^2(z)+\frac{1}{2\tilde{r}_0^3(z)}\left(\frac{4}{75}z\tilde{L}^2-\tilde{L}^2(z)\tilde{r}_0(z)+\frac{4}{75}z\tilde{r}_0^2(z)+\frac{1}{3}\tilde{r}_0^5(z)\right)\right). \tag{48}$$

Writing $\tilde{\tau}=\sqrt{\Lambda}\tau$ and $r=(1/\sqrt{\Lambda})\tilde{r}(z,\tilde{\tau})$, equations (14) and (19) reduce to

$$\left(\frac{\partial\tilde{r}}{\partial\tilde{\tau}}\right)_z^2=\frac{4z}{75\tilde{r}}\left(1+\frac{\tilde{L}^2}{\tilde{r}^2}\right)-\frac{\tilde{L}^2}{\tilde{r}^2}+2E+\frac{1}{3}\tilde{L}^2+\frac{1}{3}\tilde{r}^2 \tag{49}$$

$$\tilde{g}(z)=\left(\frac{\partial\tilde{\tau}}{\partial z}\right)_t=-\frac{\tilde{r}^2\tilde{L}}{\tilde{L}^2+\tilde{r}^2}\left(\frac{\partial}{\partial z}\right)_\tau\left(\int_{\tilde{r}(0,z)}^{\tilde{r}}\frac{\tilde{L}}{\tilde{r}^2(\partial\tilde{r}/\partial\tilde{\tau})}\mathrm{d}\tilde{r}\right). \tag{50}$$

The metric, (20), is given by $\mathrm{d}s^2=\mathrm{d}\tilde{s}^2/\Lambda$, where

$$\mathrm{d}\tilde{s}^2=\left(1+\frac{\tilde{L}^2}{\tilde{r}^2}\right)\mathrm{d}\tilde{\tau}^2-2\left(1+\frac{\tilde{L}^2}{\tilde{r}^2}\right)\tilde{g}\mathrm{d}\tilde{\tau}\mathrm{d}z-\tilde{r}^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)$$

$$-\frac{\tilde{r}^2+\tilde{L}^2}{\tilde{r}^2(1+2E)}\left(\tilde{r}'^2+2\tilde{g}\tilde{r}'\frac{\partial\tilde{r}}{\partial\tilde{\tau}}+\tilde{g}^2\left(\left(\frac{\partial\tilde{r}}{\partial\tilde{\tau}}\right)^2-(1+2E)\right)\right)\mathrm{d}z^2. \tag{51}$$

In this $'$ now denotes $(\partial/\partial z)_{\tilde{\tau}}$. This solution exhibits three types of behaviour. If $r(z,0)=r_0$, the universe is static, an Einstein cluster. If the '$-$' solution is taken and $r(0,z)<r_0$, the result is a universe which undergoes coasting expansion. If the '+' solution is used and $r(0,z)>r_0$, the universe undergoes coasting collapse. In the latter two cases, the asymptotic state is a static solution. The '+' solution is only relevant to collapse, since $E-V<0$ for r slightly less than r_0 in that case, and vice versa for the '$-$' solution.

We may write $2(E-V(r,z))=(\tilde{r}-\tilde{r}_0)^3(\tilde{r}-\tilde{r}_1)(\tilde{r}-\tilde{r}_2)/(3\tilde{r}^3)$ for the right hand side of (49), where $\tilde{r}_{1,2}=-\frac{3}{2}\tilde{r}_0\pm\sqrt{\frac{9}{4}\tilde{r}_0^2+4z\tilde{L}^2/(25\tilde{r}_0^3)}$. The solution to (49) for the expansion case, taking a big bang initial condition $r(0,z)=0$ is then given by

$$\tilde{\tau}=\frac{2\sqrt{3}\tilde{r}_0}{\tilde{r}_1-\tilde{r}_0}\left[\sqrt{\frac{(\tilde{r}_1-\tilde{r})\tilde{r}}{(\tilde{r}_0-\tilde{r})(\tilde{r}-\tilde{r}_2)}}-\sqrt{\frac{\tilde{r}_1}{\tilde{r}_0-\tilde{r}_2}}\mathbf{E}(\delta,q)\right]$$
$$-\frac{2\sqrt{3}\tilde{r}_2}{\sqrt{\tilde{r}_1(\tilde{r}_0-\tilde{r}_2)}}\left[\mathbf{F}(\delta,q)-\mathbf{\Pi}\left(\tilde{r}_0/(\tilde{r}_0-\tilde{r}_2);\delta,q\right)\right]. \tag{52}$$

Denoting the integral inside $(\partial/\partial z)$ in (50) by $B(\tilde{\tau},z)$, we have also that

$$B(\tilde{\tau},z)=\frac{2\sqrt{3}}{\tilde{r}_0(\tilde{r}_0-\tilde{r}_2)\sqrt{\tilde{r}_1(\tilde{r}_0-\tilde{r}_2)}}\left[-\tilde{r}_2\mathbf{\Pi}(1;\delta,q)+\tilde{r}_0\mathbf{F}(\delta,q)\right]. \tag{53}$$

Here $\delta=\arcsin\sqrt{((\tilde{r}_0-\tilde{r}_2)\tilde{r})/(\tilde{r}_0(\tilde{r}-\tilde{r}_2))}$ and $q=\sqrt{(\tilde{r}_0(\tilde{r}_1-\tilde{r}_2))/(\tilde{r}_1(\tilde{r}_0-\tilde{r}_2))}$. The functions $\mathbf{F}(\phi,k)$, $\mathbf{E}(\phi,k)$ and $\mathbf{\Pi}(n;\phi,k)$ are the elliptic integrals of the first, second and third kinds respectively. The second result was obtained using [25], §3.151 4 and the first by writing $\tilde{r}^2=\tilde{r}_0\tilde{r}-\tilde{r}(\tilde{r}_0-\tilde{r})$ and using [25] §3.168 46 and §3.167 12.

For the collapse case, $\tilde{r}_2<0<\tilde{r}_1<\tilde{r}_0<\tilde{r}$ and the two integrals become:

$$\tilde{\tau}=\left[\frac{2\sqrt{3}(\tilde{r}_0-\tilde{r}_1)}{\sqrt{\tilde{r}_0(\tilde{r}_1-\tilde{r}_2)}}\mathbf{\Pi}\left(\frac{\tilde{r}_0-\tilde{r}_2}{\tilde{r}_1-\tilde{r}_2};\nu,q\right)+\frac{2\sqrt{3}(\tilde{r}_0+\tilde{r}_1)}{\sqrt{\tilde{r}_0(\tilde{r}_1-\tilde{r}_2)}}\mathbf{F}(\nu,q)\right]_{\tilde{r}(0,z)}^{\tilde{r}}+\sqrt{3}\tilde{r}_0^2B(\tilde{\tau},z) \tag{54}$$

$$B(\tilde{\tau},z)=\frac{2\sqrt{3}}{\sqrt{\tilde{r}_0}(\tilde{r}_0-\tilde{r}_2)}\left[\frac{1}{\sqrt{\tilde{r}_1-\tilde{r}_2}}\mathbf{F}(\gamma,q)+\frac{\sqrt{\tilde{r}_1-\tilde{r}_2}}{\tilde{r}_0-\tilde{r}_1}\mathbf{E}(\gamma,q)\right.$$
$$\left.-\frac{\sqrt{-\tilde{r}_1\tilde{r}_2}}{\sqrt{\tilde{r}_0}(\tilde{r}_0-\tilde{r}_1)}\sqrt{\frac{(\frac{1}{\tilde{r}_1}-u)(u-\frac{1}{\tilde{r}_2})}{\frac{1}{\tilde{r}_0}-u}}\right]_{u=\frac{1}{\tilde{r}(0,z)}}^{u=\frac{1}{\tilde{r}}}. \tag{55}$$

In these, $\gamma=\arcsin\sqrt{(u-1/\tilde{r}_1)/(u-1/\tilde{r}_0)}$, $q=\sqrt{(1/\tilde{r}_0-1/\tilde{r}_2)/(1/\tilde{r}_1-1/\tilde{r}_2)}$ and $\nu=\sin^{-1}\sqrt{((\tilde{r}_1-\tilde{r}_2)(\tilde{r}-\tilde{r}_0))/((\tilde{r}_0-\tilde{r}_2)(\tilde{r}-\tilde{r}_1))}$. The result (55) is obtained by changing variable to $u=\tilde{r}^{-1}$, writing $-u=(\tilde{r}_0^{-1}-u)-\tilde{r}_0^{-1}$ and using [25] §3.131 3 and §3.133 9. Expression (54) is derived by writing $\tilde{r}^2=(\tilde{r}-\tilde{r}_0)^2+2\tilde{r}_0(\tilde{r}-\tilde{r}_0)+\tilde{r}_0^2$ and using [25] §3.147 8 and §3.167 32. In both cases, the solutions have taken a common origin of time, so that $\tau=0$ for all z at $t=0$.

Figure 3 illustrates the functions $\tilde{L}^2$, $\tilde{r}_0$ and E for the two cases. Figure 4 illustrates $r(\tilde{\tau}, z)$ for fixed values of z and $\tilde{\tau}$. The initial condition in the collapse case was chosen as $r(0, z) = z^{\frac{1}{3}}$, representing constant density. In the collapse case, expanding $\tilde{r}_0$ about $z = 1$ gives $\tilde{r}_0 > \tilde{r}_0(1)$ for $1 - z \ll 1$. This is clear from figure 3c. The region in which $\mathrm{d}\tilde{r}_0/\mathrm{d}z < 0$ is not physical, as there would have to be shell crossing. Figure 4 illustrates this, as the function $r(\tilde{\tau}, z)$ is seen to turn over for large z at later time. This problem is avoided by redefining M_{max} to be at the point where $\mathrm{d}\tilde{r}_0/\mathrm{d}z = 0$, giving $GM_{max} = 2z_{crit}/(75\sqrt{\Lambda})$. The point z_{crit} is found numerically to be at 0.84997.

Figure 3 also illustrates that for $z \approx 0$, $r_0 \propto z^{1/3}$ in the collapse case, but $r_0 \propto z$ in the expansion case. Finally, we see that these universe are not closed, since E is never equal to $-1/2$. A closed model is only possible by matching onto a closed FRW exterior.

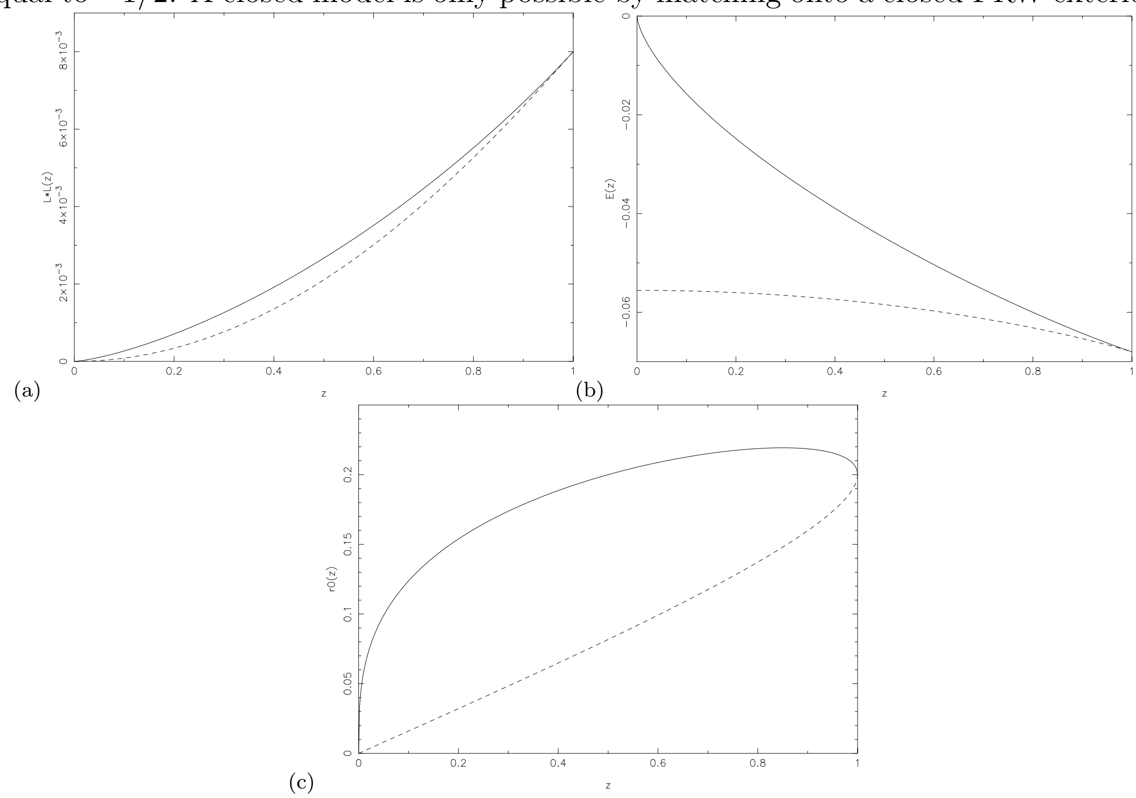

Figure 3 Plots of (a) $\tilde{L}^2(z)$, (b) $E(z)$ and (c) $\tilde{r}_0(z)$ for the repeated root solution. In each plot, the solid line is the '+' case, relevant to collapse, and the dashed line is the '-' case, relevant for expansion.

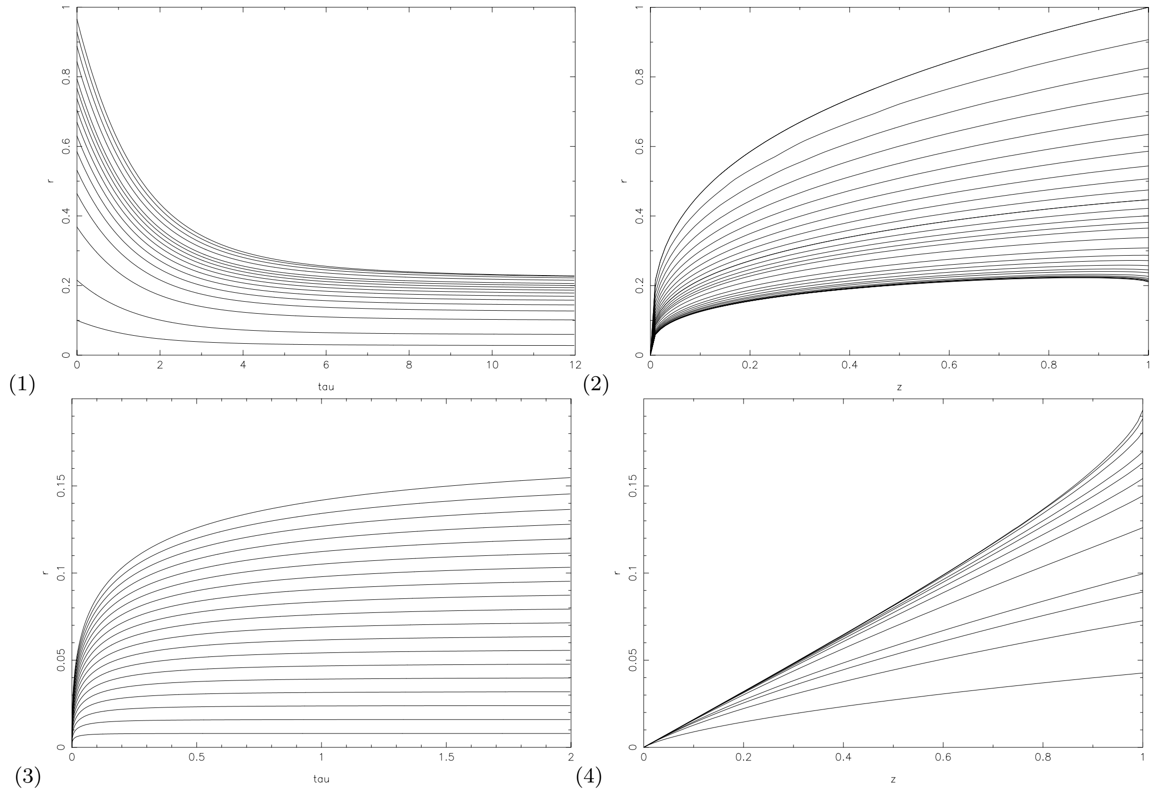

Figure 4 $r(\tilde{\tau}, z)$ at fixed z ((1), (3)) and at fixed $\tilde{\tau}$ ((2), (4)) for the repeated root solution. (1) and (2) are for the collapse case and (3) and (4) are the expansion case. The values used in (1) were $z = 0.001$, 0.01, 0.05 to 0.5 in intervals of 0.05 and 0.6 to 0.9 in intervals of 0.1. For (3), these were $z = 0.05$ to 0.95 in intervals of 0.05. The value of z increases from the bottom to the top. In (2), $\tilde{\tau} = 0.0$ to 2.8 in intervals of 0.2, and $\tilde{\tau}$ increases from top to bottom. In (3), $\tilde{\tau} = 0.01, 0.05, 0.1, 0.15, 0.4, 0.8, 1.2, 1.8, 2.5, 5.0, 10.0, 20.0$ and $\tilde{\tau}$ increases from bottom to top.

Limiting Behaviour As $\tau \to \infty$, these solutions approach the static Einstein cluster with $r = r_0(z)$. Defining $D_\pm = 2\sqrt{X}P_\pm(X)/3$, X is given as $\sqrt{X} = (3D_\pm^2 + 5)/(4D_\pm)$, whence (45)-(48) give E, L and $r = r_0$ as functions of $D_\pm$. Substituting into equation (9), and using partial fractions, ν can be evaluated as

$$\nu = \int \frac{\tilde{L}^2}{\tilde{L}^2 + \tilde{r}_0^2} \frac{\mathrm{d}\tilde{r}_0}{\tilde{r}_0} = \frac{1}{8}\ln(D-1) + \frac{1}{8}\ln(D+1) - \ln D + \frac{3}{8}\ln(3D^2+1)$$
$$- \frac{1}{\sqrt{3}}\arctan\sqrt{3}D + \frac{\sqrt{3}}{4\sqrt{5}}\arctan\left(\sqrt{\frac{3}{5}}D\right). \tag{56}$$

This gives the metric of the static system in terms of the coordinates t and D. In the '+' case, $\tilde{r}_0(z) \propto z^{\frac{1}{3}}$ near $r = 0$ and so the density $\epsilon \propto M'/(r^2 r')$ is finite everywhere. There is no singularity in the asymptotic solution and so no naked singularity forms during the collapse.

In the '-' case $\tilde{r}_0(z) \propto z$ near $z = 0$ and $\epsilon \propto z^{-2}$ diverges. Using (45)-(48) and (56), the metric near the origin is $\mathrm{d}s^2 = Az^{\frac{1}{2}}\mathrm{d}t^2 - B\mathrm{d}z^2 - \left(\frac{4}{25}z\right)^2 \mathrm{d}\Omega^2$, where A and B are positive constants. It is thus clear that for a radial photon, $\mathrm{d}z/\mathrm{d}t \propto z^{\frac{1}{4}}$ near the origin. This is integrable, so photons can escape from the central singularity - it is locally naked.

The singularity is always present, as a remnant of the big bang. The '-' situation is not appropriate to gravitational collapse, and so this naked singularity could not arise out of collapse. However, the presence of a naked singularity in a universe born from a big bang is still interesting.

In [12], taking $M = m_3R^3 + m_5R^5 + ...$ and $L^2 = l_4R^4 + l_6R^6 + ...$ near $R = 0$, they found that if $l_4 \neq 0$ no central singularity forms, but one does if $l_4 = 0$. In this example $X = L^2/(3G^2M^2)$, so the previous relations with $l_4 \neq 0$ give $X \propto R^{-2} \propto M^{-\frac{2}{3}}$ near the central singularity at $R = 0$. This is the situation in the '+' case here, when $X \propto z^{-\frac{2}{3}} \rightarrow \infty$ near $z = 0$. Once again no singularity forms under these conditions. In the $l_4 = 0$ case, $X \rightarrow$ *const.*, which is true for the '-' case here. In fact $\tilde{L}^2 \propto z^2$ near $z = 0$. The formation of the singularity in the second case and not the first is thus consistent with the conditions discussed by Harada *et al* .

5. Summary

A new approach to the Datta model of a spherical system of dust with angular momentum has been presented and investigated. Einstein's equations have been solved to give the metric in terms of two physical coordinates - the shell label and the proper time felt by the dust particles composing each shell. Conditions for regular evolution of these models have been considered and the possible types of evolution discussed. In addition to collapsing, expanding and static models, there are bouncing universes and oscillating universes, which may evolve without shell crossing. Some specific examples have been looked at in detail. The application of this approach to the formation of naked singularities has been discussed with reference to Einstein clusters and their $\Lambda \neq 0$ generalization. We find that naked singularities may form, but that these are not stable in the sense that spherical perturbations will lead to the singularity being clothed.

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Appendix A. Classification of the Potential for non-zero Λ

In the case $\Lambda = 0$, (22) tells us that $\partial V/\partial r = GM(r - r_+)(r - r_-)/r^4$, where $r_\pm = L^2\left(1 \pm \sqrt{1 - 12G^2M^2/L^2}\right)/2GM$. The potential then has 0, 1 or 2 turning points as $X = L^2/(3G^2M^2)$ is less than, equal to or greater than 4. The signs of $V(r_\pm, R)$ indicate how many times the potential cuts the r-axis. If Λ is non-zero the potential is a quintic divided by r^3. The shape of $V(r, R)$ depends on the number of turning points. The transition between the various shapes occurs when the potential has a point of inflection, i.e. when $\partial V/\partial r$ has a repeated 0. Differentiating equation (22) and requiring $\partial V/\partial r$ to have a double zero at $r = r_0$, gives the condition that $F_\pm = Q_\pm(X)/\Lambda - Y = 0$, where $X = L^2/(3G^2M^2)$, $Y = 3L^2$ and $Q_\pm(X)$ is given by:

$$Q_\pm(X) = \frac{27(4P_\pm - 3)}{640X^2P_\pm - 1200(1 + P_\pm)X + 1125}$$
$$\text{with} \;\; P_\pm = 1 \pm \sqrt{1 - \frac{15}{4X}}. \tag{A.1}$$

This repeated root is at $r_0 = 2\sqrt{XY}P_\pm(X)/3$. The functions $Q_\pm$ are defined only for $X > 3.75$, so that the square root in (A.1) is real. A physically reasonable solution must have $Y \propto L^2 \geq 0$. For $\Lambda > 0$, both solutions satisfy this for $3.75 \leq X \leq 4.0$, and Q_+ does for $X > 4.0$. If $\Lambda < 0$, only the Q_- solution can have $L^2 > 0$, and X must be greater than 4.0. Using these functions, the shape of the potential may be classified by the following scheme:

$\underline{\Lambda > 0}$:		
$X < 3.75$		one turning point only
$3.75 \leq X \leq 4.0$	$\Lambda Y < Q_-(X)$	one turning point only
	$\Lambda Y = Q_-(X)$	2 t.p.s – inflection to left
	$Q_-(X) < \Lambda Y < Q_+(X)$	three turning points
	$\Lambda Y = Q_+(X)$	2 t.p.s – inflection to right
	$\Lambda Y > Q_+(X)$	one turning point only
$4.0 < X$	$\Lambda Y < Q_+(X)$	three turning points
	$\Lambda Y = Q_+(X)$	2 t.p.s – inflection to right
	$\Lambda Y > Q_+(X)$	one turning point only
$\underline{\Lambda < 0}$:		
$X \leq 4.0$		no turning points
$X \geq 4.0$	$\Lambda Y < Q_-(X)$	no turning points
	$\Lambda Y = Q_-(X)$	one point of inflection
	$\Lambda Y > Q_-(X)$	two turning points.

The cases when the point of inflection is to the left or right of the other turning point are distinguished by comparing r_0 to $(3GM/(10\Lambda))^{\frac{1}{3}}$, which is where the second derivative of $r^3V(r,R)$ is zero. If $r_0 > (3GM/(10\Lambda))^{\frac{1}{3}}$, the repeated root is to the right of the other root, and it is to the left otherwise. This reduces to the condition $(10Q_\pm)^{-\frac{1}{3}} - 2X^{\frac{2}{3}}P_\pm(X)/3 < 0$, which is a function of X only. For Q_+, this inequality is satisfied for all X. For Q_-, it is satisfied for $X > 4.0$, but not for $3.75 < X < 4.0$.

The cases that $\partial V/\partial r$ has 3 zeros or just 1 are distinguished using the function $(\partial F_\pm/\partial c)_{a,b}$ evaluated at $F_\pm = 0$ (in this, $a = 3GM/\Lambda$, $b = 3L^2/\Lambda$ and $c = 9GML^2/\Lambda$). This function determines the direction the potential moves as X moves away from the repeated root value. In fact $(-a(\partial F_\pm/\partial c)_{a,b})_{|F_\pm=0} = X\mathrm{d}\ln Q_\pm/\mathrm{d}X + 2$, a function of X only. This function is always negative for Q_+. For Q_-, it is positive if $X > 4.0$ and negative for $3.75 < X < 4.0$.

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